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Cosmological fluctuations of a random field and radiation fluid

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A generalization of the random fluid hydrodynamic fluctuation theory due to Landau and Lifshitz is applied to describe cosmological fluctuations in systems with radiation and scalar fields. The viscous pressures, parametrized in terms of the bulk and shear viscosity coefficients, and the respective random fluctuations in the radiation fluid are combined with the stochastic and dissipative scalar evolution equation. This results in a complete set of equations describing the perturbations in both scalar and radiation fluids. These derived equations are then studied, as an example, in the context of warm inflation. Similar treatments can be done for other cosmological early universe scenarios involving thermal or statistical fluctuations.

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I. INTRODUCTION

Fluctuation and dissipation phenomena could potentially play an important role in early universe cosmology. When the matter content of the universe can be split into a subsystem interacting with a large energy reservoir, then physical processes may be represented through effective dissipation and stochastic noise terms. Various physical systems have been proposed for the early universe which are well suited for such a treatment. In particular scenarios where thermal or any statistical fluctuations seed cosmic structure fit into this category. The role of thermal fluctuations in structure formation have been considered since the early work of Peebles [1] and Harrison [2]. In recent times, thermal fluctuations continue to be seen as a possible mechanism for seeding structure formation [3–11]. One of the early models with statistical fluctuations seeding structure formation was in the context of inflationary cosmology with the warm inflation paradigm [5]. There are a variety of warm inflation models that have been developed (see [12, 13] and references therein for several examples). Moreover, many models have subsequently been developed that are very similar to the warm inflation picture, with particle production during inflation and non-vacuum density perturbations, such as non-commutative inflation [14], decaying multifield inflation models [15], trapped inflation [16], cyclic-inflation models [11, 17], axion models of inflation [18], and effective field theory models of warm inflation [19]. Thermal fluctuations have also been examined as the origin of density perturbations in bouncing universe models [7, 20], string cosmology [21, 22], loop cosmology [23], the near-Milne universe [24], a model of phase transition involving holography [25], and varying speed of light models [7]. Density perturbations created from a statistical state have been examined in the radiation dominated regime [7–11] and other excited states [26, 27]. During reheating and preheating, particle production has been shown to affect cosmological perturbations [28–30].

In all these scenarios a common feature is a sizable density of particles in the universe that is pictured in some statistical state, usually thermal. In order to thermalize, these particles must interact with one another. The short scale physics in the early universe is not directly accessible to observation today. Moreover this is a complicated many-body problem, which, similar to such problems in condensed matter systems, would be effectively intractable to exact calculation. The typical approach to such a problem is for this many-body dynamics to be characterized by one, perhaps many, microphysical scales, within which one can employ a statistical treatment and the dynamics manifests itself through dissipation processes. Associated with such effects will be corresponding stochastic forces.

A treatment involving fluctuation-dissipation dynamics can be implemented at different levels of coarse graining of the degrees of freedom. Ideally one should start with the underlying fundamental quantum field theory and proceed to coarse grain. This level of sophistication has been realized in some simple cases, most notably the Caldeirra-Leggett model for condensed matter systems [31] and this model has also been examined in a cosmological context [32]. In this

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case the quantum model can be coarse grained systematically into a stochastic langevin equation for the system, with the remaining degrees of freedom represented as a noise force and a dissipative term. In more complicated quantum field theory models, coarse graining has been done at a perturbative level [33–37]. Such approaches treat one part of the quantum field theory as a system and then integrate out the remaining fields into a heat reservoir.

In treating cosmological perturbations, the problem is a little more involved than simply deriving the stochastic evolution equation for the system. The heat bath will also have fluctuation-dissipation effects due to the action of the system as well as internal effects from within the heat bath. These effects will play a role in the cosmological perturbation equations and have consequences for the density perturbations. This is a much harder problem. Deriving the full system-heat bath dynamics from quantum field theory with all fluctuation and dissipation effects computed to our knowledge has never been achieved. An intermediate approach is to treat the heat bath within a fluid approximation which is then coupled to the system, which is treated from quantum field theory. It is at this level that the study in this paper proceeds. In this approximation, the heat bath is treated in terms of quantum fields for computing the transport and noise coefficients of the system and the heat bath itself. However, in treating the cosmological perturbations, the heat bath is then represented as a fluid. The missing step here is showing how the quantum fields that comprise the heat bath can be represented as a fluid. At an intuitive level the correspondence seems evident, but it is a difficult problem of coarse graining that goes well beyond the concerns of the specific problem being addressed in this paper. This limitation in our approach thus brings some lack of precision in formulating the dynamical problem. Nevertheless, it still captures a great deal of the physics that otherwise is completely ignored in simple mean field treatments.

Once the problem is formulated in this way, progress can be made. Hydrodynamics is a macroscopic theory describing the behavior of averaged or mean variables corresponding to the energy density, pressure, fluid velocity and so on. As such, the microscopic physics become manifest in the form of dissipative terms corresponding to the transport coefficients, like bulk and shear viscosities. But from a fluctuation-dissipation stand point, these must also be related to stochastic fluctuations as well. Landau and Lifshitz were the first to propose a fluctuating hydrodynamics theory, where random fluxes are added to the usual hydrodynamic equations, with two-point correlation functions related with the transport coefficients through fluctuation-dissipation relations [38]. The Landau-Lifshitz fluctuating hydrodynamics theory was later refined and put in firm theoretical grounds by the work of Fox and Uhlenbeck [39] and extended to the relativistic fluids by Zimdahl [40] (see also [41]).

Aside for warm inflation, most cosmological models involving thermal or statistical fluctuations have been examined at only a mean field level, where fluctuation-dissipation effects have not been treated. As such, important information is ignored about how the short scale physics affects the large scale physics. Calculations in warm inflation have shown that current precision from CMB data demands a treatment beyond a mean field level, and requires account for fluctuation and dissipation effects. This lesson probably also carries over for other scenarios involving thermal fluctuations.

In this paper, we will study the density perturbations spectra in terms of the coupled set of radiation equations describing the random radiation fluid equations and the stochastic equation for a scalar field. We believe this is the first study of such a system. The study of cosmological perturbations making use of the relativistic version of the fluctuation hydrodynamics theory of Landau and Lifshitz has already been done before by Zimdahl in [42]. Some papers have treated the density perturbations in a system of a scalar field with dissipation coupled to a radiation fluid [43] as well as affects of viscosity within the radiation fluid [44, 45]. However no work has treated in addition the corresponding noise forces that accompany dissipation and viscosity. Our treatment in this paper includes all these effects and can be applied to problems in cosmology involving a scalar field coupled to other systems, such as in inflationary cosmology, cosmic phase transitions, reheating, curvaton decay etc... Often, in addition of including the scalar field dynamics, we also have a mechanism by which the radiation bath is generated and maintained through particle production due to the decay of the scalar field. We analyze in detail not only the interplay of the different dissipation terms, from the scalar field and the bulk and shear viscosities of the radiation fluid, but also the effect of the respective noise terms, connected with the dissipation and viscosity terms by the dissipation-fluctuation relations.

This paper is organized as follows. In Sec. II we introduce the relativistic fluctuating hydrodynamics built from the original version due to Landau and Lifshitz. In Sec. III, guided by the equivalence principle, we extend the relativistic fluctuating hydrodynamic equations for the radiation bath for the cosmological context. The equations are coupled with those of the inflaton field as appropriate in dissipative environments, like in the warm inflation scenario, and the perturbation equations constructed. In Sec. IV we give the general expressions for the dissipative and viscosity coefficients we will be considering along this work. In Sec. V, the full cosmological perturbations are studied numerically and results for the curvature power spectrum for perturbations presented. The effects of both bulk and shear viscosities are analyzed. Finally, in Sec. VI we give our concluding remarks.

II. FLUCTUATIONS IN FLAT SPACETIME

We are interested primarily in situations with a radiation fluid that is close to being in thermal equilibrium at some local temperature T , and the fluid is hot enough to be treated as classical and relativistic. Quantum statistical mechanical fluctuations in such a radiation fluid can be described using Landau's theory of random fluids [38], where the deterministic equations of fluid dynamics are replaced by a system of equations with stochastic source terms. The fluid approximation is maintained by microscopic interactions, with small departures from equilibrium which cause both fluctuations and dissipation. The fluctuations of the fluid reach a balance between the effects of the source and the dissipation terms. Fixing the statistical properties of the noise terms to ensure that stochastic averages of fluid variables reproduce the statistical ensemble averages leads to the fundamental fluctuation-dissipation relation.

In this work we will be using the first-order (or Eckart) dissipative relativistic hydrodynamics. Even though the first-order formalism is known to have problems concerning causality, stability and in general they do not have a well-posed initial value formulation, the first order formalism of hydrodynamics is still simpler and more immediate to study in the context of cosmological perturbations than the second and higher order formalisms of hydrodynamics (for a recent review, see, e.g., ref. [46] and also ref. [47] for the case of including fluctuations). Furthermore, the use of the Eckart first order formalism can be justified on general grounds for small enough radiation bath relaxation times [45], which is one of the conditions required to justify a close to equilibrium thermal bath, and assumed in the application performed in section V.

Consider a relativistic fluid with energy density $\rho^{(f)}$ and pressure $p^{(f)}$ in which conserved particle numbers are absent or negligible, and the 4-velocity $u^{a(f)}$ is the velocity of energy transport. Random sources and dissipative stresses are introduced via a stress term Π_{ab} in the stress-energy tensor,

$$T^{(f)}_{ab} = (p^{(f)} + \rho^{(f)}) u_a^{(f)} u_b^{(f)} + p^{(f)} g_{ab} + \Pi_{ab}, \quad (2.1)$$

where indices $a, b \dots$ denote spacetime components. In Landau's theory, dissipation is governed by constitutive relations for shear viscosity η_s and bulk viscosity η_b whilst fluctuations are generated by a Gaussian noise term Σ_{ab} . In a comoving frame where the spatial components $u_i^{(f)} = 0$ and the time component $u_0^{(f)} = -1$, the non-vanishing shear terms are

$$\Pi_{ij} = - \left(\eta_s \nabla_i u_j^{(f)} + \eta_s \nabla_j u_i^{(f)} + (\eta_b - \frac{2}{3} \eta_s) \delta_{ij} \nabla_k u^{(f)k} \right) - \Sigma_{ij}, \quad (2.2)$$

where ∇_i denotes a spatial derivative. The correlation functions of the stochastic noise term Σ_{ij} are assumed to be local and determined by the fluctuation-dissipation relation,

$$\langle \Sigma_{ij}(x, t) \Sigma_{kl}(x', t') \rangle = 2T \left(\eta_s \delta_{ik} \delta_{jl} + \eta_s \delta_{il} \delta_{jk} + (\eta_b - \frac{2}{3} \eta_s) \delta_{ij} \delta_{kl} \right) \delta^{(3)}(x - x') \delta(t - t'). \quad (2.3)$$

This will be explored further in Sect. IIC. Landau's theory can be used reliably for small departures from a stable underlying fluid flow. We shall be concerned mostly with small density, pressure and 3-velocity fluctuations $\delta \mathbf{u}^{(f)}$ of a radiation fluid in an inertial frame with background density $\rho^{(f)}$ and pressure $p^{(f)}$. For example, the momentum conservation equation obtained using the vanishing divergence of the stress-energy tensor outlined above is,

$$(p^{(f)} + \rho^{(f)}) \delta \dot{\mathbf{u}}^{(f)} + \nabla \delta p + \dot{p}^{(f)} \delta \mathbf{u}^{(f)} = \eta_s \nabla^2 \delta \mathbf{u}^{(f)} + \frac{1}{3} \eta_s \nabla \nabla \cdot \delta \mathbf{u}^{(f)} + \eta_b \nabla \nabla \cdot \delta \mathbf{u}^{(f)} + \nabla \cdot \Sigma. \quad (2.4)$$

This can be recognized as the perturbed Navier-Stokes momentum conservation equation with a stochastic source term. The solutions to the stochastic fluid equations can be used to follow the evolution of quantities such as the density perturbations,

$$\langle \delta \rho^{(f)}(x, t) \delta \rho^{(f)}(x', t) \rangle, \quad (2.5)$$

by taking a stochastic average. Without the theory of random fluids, we would only have knowledge of the equilibrium values of the density fluctuations.

A. Relativistic fluids coupled to a scalar field

Our aim is to couple this radiation fluid to a scalar field. The behavior of a relativistic scalar field in flat spacetime interacting with radiation can be analyzed using non-equilibrium quantum field theory [48]. When the small-scale behavior of the fields is averaged out, the scalar field fluctuations, like the fluid fluctuations, can be described by a stochastic system whose evolution is determined by a Langevin equation [52]. For a weakly interacting radiation gas, the dissipation and noise terms in the Langevin equation can be approximated by local expressions. This is the case we will consider here. The Langevin equation for a scalar field with thermodynamic potential $\Omega(\phi, T)$ and damping coefficient $\Upsilon(\phi, T)$ is then [35]

$$-\square\phi(x, t) + \Upsilon\dot{\phi}(x, t) + \Omega_{,\phi} = (2\Upsilon T)^{1/2}\xi^{(\phi)}(x, t), \quad (2.6)$$

where \square is the flat spacetime d'Alembertian and $\xi^{(\phi)}$ is a stochastic source. The probability distribution of the source term will be approximated by a localized gaussian distribution with correlation function [35, 36],

$$\langle \xi^{(\phi)}(x, t) \xi^{(\phi)}(x', t') \rangle = \delta^{(3)}(x - x') \delta(t - t'). \quad (2.7)$$

The Langevin equation applies when the surrounding radiation is at rest. For a fluid in uniform motion with 4-velocity $u^{a(f)}$ we would need to choose the Lorentz frame to be the rest frame of the fluid. This can be expressed in covariant form by replacing $\dot{\phi}$ in the dissipation term by a fluid derivative,

$$D\phi = u^{a(f)} \nabla_a \phi. \quad (2.8)$$

The dissipation results in a transfer of energy and momentum from the scalar field to the radiation which needs to be included in the fluid equations.

Energy and momentum transfer can be tracked by considering the divergence of the stress-energy tensor. We combine the fluid and scalar contributions into a unified stress-energy tensor given by

$$T_{ab} = Ts u_a^{(f)} u_b^{(f)} - \Omega g_{ab} + \nabla_a \phi \nabla_b \phi - \frac{1}{2} (\nabla \phi)^2 g_{ab} + \Pi_{ab}, \quad (2.9)$$

where Π_{ab} is orthogonal to the fluid velocity. We have introduced the entropy density s , defined by the thermodynamic relation

$$s = -\Omega_{,T}. \quad (2.10)$$

If $s_{,\phi} \equiv 0$, then the thermodynamic potential Ω splits into an effective potential $V(\phi)$ depending only on ϕ and a radiation term depending only on T ,

$$\Omega = V(\phi) - p^{(f)}(T). \quad (2.11)$$

In this case, $s \equiv s(T)$ and the fundamental thermodynamic relation implies that $Ts = \rho^{(f)} + p^{(f)}$ allowing us to separate off the fluid stress-energy tensor $T^{(f)}_{ab}$ given in Eq. (2.1). This separation into fluid and scalar field terms is not possible in general, but a partial separation can be seen in the divergence of the stress-energy tensor,

$$\nabla^b T_{ab} = \left(D(Ts) + \nabla_b u^{b(f)} \right) u_a^{(f)} + s \nabla_a T + \nabla^b \Pi_{ab} + (\square\phi - \Omega_{,\phi}) \nabla_a \phi. \quad (2.12)$$

The first three terms represent the field equations for the fluid in the absence of the scalar field and they can be separated from the remaining terms by defining fluxes $Q_a^{(f)}$ and $Q_a^{(\phi)}$ by,

$$Q_a^{(f)} = \left(D(Ts) + \nabla_b u^{b(f)} \right) u_a^{(f)} + s \nabla_a T + \nabla^b \Pi_{ab}, \quad (2.13)$$

$$Q_a^{(\phi)} = (\square\phi - \Omega_{,\phi}) \nabla_a \phi. \quad (2.14)$$

Using the Langevin Eq. (2.6) for the scalar field, we obtain that

$$Q_a^{(\phi)} = \Upsilon(D\phi)\nabla_a\phi - (2\Upsilon T)^{1/2}\xi^{(\phi)}\nabla_a\phi. \quad (2.15)$$

Energy-momentum conservation $\nabla^b T_{ab} = 0$ results in a set of fluid equations,

$$\left(D(Ts) + \nabla_b u^{(f)b}\right) u^{(f)}_a + s\nabla_a T + \nabla^b \Pi_{ab} = -Q_a^{(\phi)}. \quad (2.16)$$

Therefore, the flux $Q_a^{(\phi)}$ describes the transfer of energy and momentum to the fluid equations.

As a matter of fact, Eq. (2.15) is not the most general expression which we can obtain for the energy transfer. We might also consider adding a stochastic energy flux term \mathbf{P} to the stress energy tensor, rather like the stochastic stress term Σ_{ij} which we had in Eq. 2.2, so that the stress energy tensor becomes

$$T_{ab} = Ts u^{(f)}_a u^{(f)}_b - \Omega g_{ab} + \nabla_a \phi \nabla_b \phi - \frac{1}{2}(\nabla\phi)^2 g_{ab} + \Pi_{ab} + 2u^{(f)}_a P_b. \quad (2.17)$$

This modifies the energy transfer vector $Q^{a(\phi)}$,

$$Q_a^{(\phi)} = \Upsilon(D\phi)\nabla_a\phi - (2\Upsilon T)^{1/2}\xi^{(\phi)}\nabla_a\phi + \nabla^b \left(2u^{(f)}_a P_b\right). \quad (2.18)$$

The time component represents energy transfer,

$$Q_0^{(\phi)} = \Upsilon(D\phi)\dot{\phi} - (2\Upsilon T)^{1/2}\xi^{(\phi)}\dot{\phi} - \nabla \cdot \mathbf{P}. \quad (2.19)$$

The simplest possibility is simply $P_a = 0$, but an interesting alternative is to impose the condition that the energy flux is independent of $\xi^{(\phi)}$, by setting

$$\nabla \cdot \mathbf{P} = -(2\Upsilon T)^{1/2}\xi^{(\phi)}\dot{\phi}. \quad (2.20)$$

In this case \mathbf{P} has to be included in the momentum flux $\mathbf{Q}^{(\phi)}$. The calculations in later sections will consider both of these possibilities.

B. Perturbation theory

We perturb the fluid quantities and the scalar field, replacing $\rho^{(f)}$ by $\rho^{(f)} + \delta\rho^{(f)}$ and so on, and taking the backgrounds to be homogeneous with vanishing velocity. From this point on we use the indices $i, j \dots$ to denote the spatial coordinate frame in which the background fluid is at rest. The non-vanishing components of the stress tensor Π_{ab} are given by the constitutive relations for shear and bulk viscosity as well as the random noise term Σ_{ij} generating the fluctuations,

$$\Pi_{ij} = -\left(\eta_s \nabla_i \delta u^{(f)}_j + \eta_s \nabla_j \delta u^{(f)}_i + \left(\eta_b - \frac{2}{3}\eta_s\right)\delta_{ij} \nabla_k \delta u^{(f)k}\right) - \Sigma_{ij}. \quad (2.21)$$

The noise term is taken to be gaussian with the correlation function (2.3). The first-order fluid equations obtained from energy-momentum conservation $\nabla^b T_{ab} = 0$ using the stress-energy tensor (2.9) are then

$$T\delta\dot{s} + \dot{s}\delta T + Ts\nabla \cdot \delta\mathbf{u}^{(f)} = -\delta Q^{(\phi)}, \quad (2.22)$$

$$\{Ts\delta\mathbf{u}^{(f)}\} \cdot + \nabla(s\delta T) - \eta_s \nabla^2 \delta\mathbf{u}^{(f)} - \left(\eta_b + \frac{1}{3}\eta_s\right)\nabla\nabla \cdot \delta\mathbf{u}^{(f)} = -\delta\mathbf{Q}^{(\phi)} + \nabla \cdot \Sigma, \quad (2.23)$$

where boldface denotes spatial vectors and $\delta Q^{(\phi)} = \delta Q^{(\phi)0} = -\delta Q_0^{(\phi)}$. Comparison with the random fluid Eq. (2.4) suggests that we should identify the fluid density and pressure perturbations as

$$\delta\rho^{(f)} = T \delta s, \quad (2.24)$$

$$\delta p^{(f)} = s \delta T. \quad (2.25)$$

The fluctuations $\delta\rho^{(f)}$, $\delta p^{(f)}$ and $\delta\phi$ are obtained from just two thermodynamical degrees of freedom ϕ and T , so one of the fluctuations is dependent on the other two, the natural choice being the pressure perturbation. By setting $\delta s = s_{,\phi}\delta\phi + s_{,T}\delta T$ in (2.24), we arrive at

$$\delta p^{(f)} = c_s^2(\delta\rho^{(f)} - T s_{,\phi}\delta\phi), \quad (2.26)$$

where the sound speed $c_s^2 = s/(T s_{,T})$. Differentiating Eqs. (2.24) and (2.25), we also have

$$T\delta\dot{s} + \delta T\dot{s} = \delta\dot{\rho}^{(f)} + s_{,\phi}\delta q, \quad (2.27)$$

where we have defined

$$\delta q = \dot{\phi}\delta T - \dot{T}\delta\phi. \quad (2.28)$$

The fluid equations can then be re-written in terms of the density and scalar field fluctuations,

$$\delta\dot{\rho}^{(f)} + (\rho^{(f)} + p^{(f)}) \nabla \cdot \delta\mathbf{u}^{(f)} + s_{,\phi}\delta q = -\delta Q^{(\phi)}, \quad (2.29)$$

$$\{(\rho^{(f)} + p^{(f)}) \delta\mathbf{u}^{(f)}\} \cdot + \nabla \delta p^{(f)} - \eta_s \nabla^2 \delta\mathbf{u}^{(f)} - \left(\eta_b + \frac{1}{3}\eta_s\right) \nabla \nabla \cdot \delta\mathbf{u}^{(f)} = -\delta\mathbf{Q}^{(\phi)} + \nabla \cdot \Sigma. \quad (2.30)$$

When $s_{,\phi} \equiv 0$, then $\delta p^{(f)} = c_s^2 \delta\rho^{(f)}$ and the δq term drops out of the fluid equations. In this case the equations become perturbed versions of the relativistic Navier-Stokes equations with stochastic source terms.

Since there are no sources of vorticity at linear order, we can introduce scalar velocity perturbations through

$$\delta\mathbf{u}^{(f)} = \nabla \delta v^{(f)}, \quad \delta\mathbf{Q}^{(\phi)} = \nabla \delta J^{(\phi)}. \quad (2.31)$$

The fluid perturbations for potential flow satisfy

$$\delta\dot{\rho}^{(f)} + (\rho^{(f)} + p^{(f)}) \nabla^2 \delta v^{(f)} + s_{,\phi}\delta q = -\delta Q^{(\phi)}, \quad (2.32)$$

$$\{(\rho^{(f)} + p^{(f)}) \delta v^{(f)}\} \cdot + \delta p^{(f)} - \eta' \nabla^2 \delta v^{(f)} = -\delta J^{(\phi)} + (2\eta' T)^{1/2} \xi^{(f)}, \quad (2.33)$$

where $\delta p^{(f)}$ is given by Eq. (2.26) and we have defined η' as the combination of viscosity coefficients:

$$\eta' = \frac{4}{3}\eta_s + \eta_b. \quad (2.34)$$

Using Eq. (2.3), the noise source $\xi^{(f)} = \nabla^{-2} \nabla^i \nabla^j \Sigma_{ij}$ has correlation function

$$\langle \xi^{(f)}(x, t) \xi^{(f)}(x', t') \rangle = \delta^{(3)}(x - x') \delta(t - t'). \quad (2.35)$$

The new feature of these equations is that they combine the random fluid with the exchange of energy and momentum to the scalar field, represented by the flux terms $\delta Q^{(\phi)}$ and $\delta\mathbf{Q}^{(\phi)}$. For a homogeneous background scalar field, the perturbation of Eq. (2.18) shows that

$$\delta Q^{(\phi)} = -\delta\Upsilon \dot{\phi}^2 - 2\Upsilon \dot{\phi} \delta\dot{\phi} + (2\Upsilon T)^{1/2} \dot{\phi} \xi^{(\phi)} + \nabla \cdot \mathbf{P} \quad (2.36)$$

$$\delta J^{(\phi)} = \Upsilon \dot{\phi} \delta\phi + \nabla^{-2} \nabla \cdot \dot{\mathbf{P}}. \quad (2.37)$$

We shall take

$$\mathbf{P} = -C_P(2\Upsilon T)^{1/2}\dot{\phi}\nabla^{-2}\nabla\xi^{(\phi)}. \quad (2.38)$$

The two cases $C_P = 0$ and $C_P = 1$ govern whether the noise source $\xi^{(\phi)}$ appears in the energy flux or in the momentum flux. Both cases will be considered in our numerical analysis to be performed in Sec. V. The procedure here assumes a linear transfer of energy from the ϕ -system to the radiation fluid, so that the ϕ noise term at some mode \mathbf{k} transfers energy into mode \mathbf{k} of the radiation fluid. In a quantum field theory the radiation fluid would be associated with the effective quadratic parts of the light fields in the system. In general there will be nonlinear terms transferring energy between the ϕ -field and this fluid. Thus, in associating the hydrodynamic approximation developed in this paper to an underlying quantum field theory system, this possibility of nonlinear couplings must be considered, though we will not develop this point any further in this paper.

C. Fluctuation-dissipation relations

We finish this section with a discussion of the fluctuation-dissipation relations to verify that the stochastic average

$$\langle\delta\rho^{(f)}(x,t)\delta\rho^{(f)}(x',t)\rangle, \quad (2.39)$$

reproduces the quantum-statistical ensemble average on time-independent backgrounds. This is expected on general grounds, but the derivation for relativistic fields is less well known than the non-relativistic case and the density correlations will be useful later. The thermal ensemble averages can be obtained using standard thermodynamical arguments, or by using thermal quantum field theory (see [8] for an example). These thermodynamic results have also been used in cosmological settings, e.g. by [8, 11].

We disconnect the scalar field by setting $Q_a^{(\phi)} = s_{,\phi} = 0$ and take the background density and pressure to be constant. This allows Fourier decomposition with

$$\delta\rho^{(f)}(k,\omega) = \int dt d^3x \delta\rho^{(f)}(x,t) e^{i(k\cdot x - \omega t)}, \quad (2.40)$$

$$\delta v^{(f)}(k,\omega) = \int dt d^3x \delta v^{(f)}(x,t) e^{i(k\cdot x - \omega t)}. \quad (2.41)$$

On substituting these transforms into Eqs. (2.29) and (2.30), the fluctuations satisfy

$$\begin{pmatrix} i\omega & -(1+c_s^2)\rho^{(f)}k^2 \\ c_s^2 & i\omega(1+c_s^2)\rho^{(f)} + k^2\eta' \end{pmatrix} \begin{pmatrix} \delta\rho^{(f)} \\ \delta v^{(f)} \end{pmatrix} = (2\eta'T)^{1/2} \begin{pmatrix} 0 \\ \xi^{(f)} \end{pmatrix}. \quad (2.42)$$

The solution for the density fluctuation is

$$\delta\rho^{(f)}(k,\omega) = G(k,\omega) k^2 (2\eta'T)^{1/2} \xi^{(f)}, \quad (2.43)$$

with the Green function

$$G(k,\omega) = [(\gamma k^2 - i(\omega - c_s k))(\gamma k^2 - i(\omega + c_s k)) - \gamma^2 k^4]^{-1}, \quad (2.44)$$

and $\gamma = \eta'/2(1+c_s^2)\rho^{(f)}$. After using the noise correlation function (2.35), the density correlation functions become

$$\langle\delta\rho^{(f)}(k,t)\delta\rho^{(f)}(k',t)\rangle = \int \frac{d\omega}{2\pi} |G(k,\omega)|^2 (2\eta'T) k^4 (2\pi)^2 \delta^{(3)}(k+k'). \quad (2.45)$$

In the low damping regime $\gamma k \ll c_s$, the integration gives

$$\langle\delta\rho^{(f)}(k,t)\delta\rho^{(f)}(k',t)\rangle \approx \frac{1+c_s^2}{c_s^2} T \rho^{(f)} (2\pi)^2 \delta^{(3)}(k+k'). \quad (2.46)$$

For comparison, statistical mechanics relates the fluctuations at temperatures large enough to ignore quantum effects to the entropy density s [8],

$$\langle \delta\rho^{(f)}(k, t) \delta\rho^{(f)}(k', t) \rangle_{\text{sm}} \approx T^3 \frac{\partial s}{\partial T} (2\pi)^2 \delta^{(3)}(k + k'). \quad (2.47)$$

In the case where the density depends only on temperature, we have

$$\rho^{(f)} = aT^{1+1/c_s^2}, \quad s = a(1 + c_s^2)T^{1/c_s^2}. \quad (2.48)$$

It follows that

$$\langle \delta\rho^{(f)}(k, t) \delta\rho^{(f)}(k', t) \rangle_{\text{sm}} \approx \frac{1 + c_s^2}{c_s^2} T \rho^{(f)} (2\pi)^2 \delta^{(3)}(k + k'). \quad (2.49)$$

Equations (2.46) and (2.49) agree, confirming that the coefficient of the noise term was chosen correctly.

The fluctuation-dissipation relations for the scalar field can be obtained by following a similar route. We take a constant background scalar field and consider the fluctuations $\delta\phi$. Their Fourier transforms satisfy

$$\delta\phi(k, \omega) = G(k, \omega) (2\Upsilon T)^{1/2} \xi^{(\phi)}, \quad (2.50)$$

where the Green function is

$$G(k, \omega) = (k^2 - \omega^2 - i\Upsilon\omega + m^2)^{-1}, \quad (2.51)$$

and $m^2 = V_{,\phi\phi}$. Following the same steps as above, with $\Upsilon \ll k$, these give

$$\langle \delta\phi(k, t) \delta\phi(k', t) \rangle \approx \frac{T}{\omega_k^2} (2\pi)^2 \delta^{(3)}(k + k'), \quad (2.52)$$

where $\omega_k^2 = k^2 + m^2$. This is the correct statistical mechanical result, telling us that the oscillator modes with energy $\omega_k^2 \delta\phi^2$ have an average energy T in the classical regime $\omega_k \ll T$. In the quantum regime $\omega_k \gg T$, we would have

$$\langle \delta\phi(k, t) \delta\phi(k', t) \rangle \approx \frac{1}{2\omega_k} (2\pi)^2 \delta^{(3)}(k + k'). \quad (2.53)$$

This result can be obtained by following the general prescription (see, e.g., [49] where this is explicitly derived) of inserting the factor $(\omega/2T) \cosh(\omega/2T)$ into the Fourier transform of the noise correlation (2.7).

III. COSMOLOGICAL PERTURBATIONS

In this section we shall describe the effects of fluid and scalar field fluctuations in a cosmological setting where the background spacetime describes a homogeneous, isotropic and spatially flat universe. We assume the fluid to be highly relativistic, such as we might expect in the very early universe. The main dissipative mechanisms are the energy loss by the scalar field and viscosity in the radiation fluid. Each of these is associated with a stochastic source term with correlation functions determined by the fluctuation-dissipation relation. We shall take the damping terms and the correlation functions to have a local form, allowing us to apply the equivalence principle.

Our gauge-ready formalism for cosmological perturbations follows Hwang and Noh [43]. The spacetime metric for a scalar-type of perturbation is given by

$$ds^2 = -(1 + 2\alpha)dt^2 - 2\beta_{,i}dt dx^i + a^2 (\delta_{ij}(1 + 2\varphi) + 2\gamma_{,ij}) dx^i dx^j, \quad (3.1)$$

where $a(t)$ is the scale factor and $H = \dot{a}/a$ defines the background expansions rate. Physical combinations of the metric perturbations which will be useful later on are the shear χ and perturbed expansion rate κ , given by

$$\chi = a(\beta + a\dot{\gamma}), \quad (3.2)$$

$$\kappa = 3H\alpha - 3\dot{\phi} - \nabla^2\chi. \quad (3.3)$$

The background Laplacian denotes the combination $\nabla^2 = a^{-2}\delta^{ij}\nabla_i\nabla_j$, where ∇_i is the derivative with respect to x^i . Note the factor of a^{-2} here, and that ∇^2 is the covariant Laplacian for the spatial metric $g_{ij} = a^2\delta_{ij}$.

The stress-energy tensor is conveniently expressed as,

$$T_{ab} = (\rho + p)n_a n_b + pg_{ab} + n_a q_b + n_b q_a + \Pi_{ab}, \quad (3.4)$$

where q_a and the trace-free tensor Π_{ab} are orthogonal to the unit vector n_a . We shall take n^a to be the unit normal to the constant-time surfaces. For scalar perturbations, we replace ρ by $\rho + \delta\rho$, p by $p + \delta p$ and define δv and $\delta\Pi$ by

$$q_i = (\rho + p)\nabla_i\delta v, \quad \delta\Pi_{ij} = \nabla_i\nabla_j\delta\Pi - \frac{1}{3}g_{ij}\nabla^2\delta\Pi, \quad (3.5)$$

The perturbed Einstein equations in gauge-ready form are then [43]

$$\nabla^2\varphi + H\kappa = -4\pi G\delta\rho, \quad (3.6)$$

$$\kappa + \nabla^2\chi = -12\pi G(\rho + p)\delta v, \quad (3.7)$$

$$\dot{\chi} + H\chi - \alpha - \varphi = 8\pi G\delta\Pi, \quad (3.8)$$

$$\dot{\kappa} + 2H\kappa + \nabla^2\alpha - 3(\rho + p)\alpha = 4\pi G(\delta\rho + 3\delta p). \quad (3.9)$$

Diffeomorphism invariance allows us to fix two of the independent variables. At least two further equations are required, and these come from considering the matter sector, which in our case consists of the radiation fluid and the scalar field.

A. Fluid and scalar perturbations

The stress-energy tensor can be expressed in the velocity frame we used earlier in Eq. (2.9),

$$T_{ab} = Ts u_a^{(f)} u_b^{(f)} - \Omega g_{ab} + \Pi_{ab}^{(f)} + \nabla_a\phi\nabla_b\phi - \frac{1}{2}(\nabla\phi)^2 g_{ab}. \quad (3.10)$$

We have taken the energy flux $\mathbf{P} = 0$ to simplify the discussion, but non-vanishing energy flux can easily be accommodated. By comparing the two forms of the stress-energy tensor (3.4) and (3.10) on homogeneous backgrounds with vanishing fluid velocity we find the relations,

$$\rho = Ts + \frac{1}{2}\dot{\phi}^2 + \Omega, \quad (3.11)$$

$$p = \frac{1}{2}\dot{\phi}^2 - \Omega + \Pi^{(f)}, \quad (3.12)$$

where $\Pi^{(f)} = \Pi^{(f)}_i{}^i$. For the fluctuations, comparing the first-order perturbations of the two stress-energy tensors gives

$$\delta\rho = \delta\rho^{(f)} + \dot{\phi}(\delta\dot{\phi} - \alpha\dot{\phi}) + \Omega_{,\phi}\delta\phi, \quad (3.13)$$

$$\delta p = \delta p^{(f)} + \dot{\phi}(\delta\dot{\phi} - \alpha\dot{\phi}) - \Omega_{,\phi}\delta\phi, \quad (3.14)$$

$$(\rho + p)\delta v = Ts\delta v^{(f)} - \dot{\phi}\delta\phi, \quad (3.15)$$

where $\delta\rho^{(f)} = T\delta s$ and $\delta u_i^{(f)} = \nabla_i\delta v^{(f)}$ as before. (See below for $\delta p^{(f)}$.)

B. Fluid equations

The fluid equations obtained from the stress-energy tensor (3.10) are

$$T Ds + Ts \nabla_a u^{a(f)} + \Pi^{(f)ab} \nabla_a u_b^{(f)} = -Q^{(\phi)}, \quad (3.16)$$

$$Ts Du_a^{(f)} + s h_a^b \nabla_b T + h_{ac} \nabla_b \Pi^{(f)bc} = -h_{ac} Q^{(\phi)c}. \quad (3.17)$$

where $h_{ab} = g_{ab} + u_a^{(f)} u_b^{(f)}$ and D is the comoving derivative as before. Guided by the equivalence principle, we add dissipation and noise sources to the shear stress $\Pi^{(f)}_{ab}$ to reproduce the flat-spacetime limit Eq. (2.2),

$$\Pi^{(f)}_{ab} = -2\eta_s \sigma_{ab} - \eta_b h_{ab} \nabla^c u_c^{(f)} - \Sigma_{ab}. \quad (3.18)$$

The first term relates the shear stress to the rate-of-strain tensor σ_{ab} ,

$$\sigma_{ab} = h_{(a}^c h_{b)}^d \nabla_c u_d^{(f)} - \frac{1}{3} h_{ab} \nabla^c u_c^{(f)}. \quad (3.19)$$

Note that the bulk viscosity terms behave like a contribution to the pressure $p^{(f)}$.

We are ready to expand these equations to first order in perturbations theory about homogeneous backgrounds. The background fluid equation from Eq. (3.16) is

$$T\dot{s} + 3H(Ts - 3H\eta_b) = \Upsilon\dot{\phi}^2. \quad (3.20)$$

At first order in perturbation theory, using the metric (3.1), we find the velocity expansion,

$$\delta(\nabla^c u_c^{(f)}) = \nabla^2 \delta v^{(f)} - \kappa, \quad (3.21)$$

and the strain tensor

$$\sigma_{ij} = \nabla_i \nabla_j \sigma - \frac{1}{3} g_{ij} \nabla^2 \sigma, \quad \sigma = \delta v^{(f)} + \chi. \quad (3.22)$$

We can also modify the pressure to absorb the bulk viscosity, by defining

$$\delta p^{(f)} = s\delta T - 3H\delta\eta_b = c_s^2(\delta\rho^{(f)} - Ts_{,\phi}\delta\phi) - 3H\delta\eta_b, \quad (3.23)$$

after taking into account Eq. (2.26). The fluid equations (3.16) and (3.17) expanded to first order with the metric (3.1) become

$$\delta\dot{\rho}^{(f)} - \alpha T\dot{s} + 3H(\delta\rho^{(f)} + \delta p^{(f)} - \eta_b \kappa) + (Ts - 3H\eta_b)(\nabla^2 \delta v^{(f)} - \kappa) + s_{,\phi}\delta q = -\delta Q^{(\phi)}, \quad (3.24)$$

$$a^{-3}\{a^3(Ts - 3H\eta_b)\delta v^{(f)}\} + \alpha(Ts - 3H\eta_b) + \delta p^{(f)} - \eta_b \kappa - \eta' \nabla^2(\delta v^{(f)} + \chi) = -\delta J^{(\phi)} + (2\eta' T)^{1/2} \xi^{(f)}. \quad (3.25)$$

These equations reduce to the previous set of equations (2.32) and (2.33) in flat space if we substitute $Ts = \rho^{(f)} + p^{(f)}$, although it is often advantageous to work with s and T rather than $\rho^{(f)}$ and $p^{(f)}$.

The correlation function for the stochastic sources has to be corrected to account for the scaling between comoving coordinates x^i and inertial frame coordinates ax^i , resulting in a factor a^{-3} ,

$$\langle \xi^{(f)}(x, t) \xi^{(f)}(x', t') \rangle = a^{-3} \delta^{(3)}(x - x') \delta(t - t'). \quad (3.26)$$

The energy and momentum transfer terms are given as before by perturbing Eq. (2.15),

$$\delta Q^{(\phi)} = -\delta\Upsilon\dot{\phi}^2 - 2\Upsilon\dot{\phi}(\delta\dot{\phi} - \alpha\dot{\phi}) + (2\Upsilon T)^{1/2}\dot{\phi}\xi^{(\phi)} + \nabla \cdot \mathbf{P}, \quad (3.27)$$

$$\delta J^{(\phi)} = \Upsilon\dot{\phi}\delta\phi + \nabla^{-2}\nabla \cdot (\dot{\mathbf{P}} + 4H\mathbf{P}), \quad (3.28)$$

where we have allowed for the possibility of modifying the stress energy tensor by including a stochastic energy flux term \mathbf{P} .

The fluid equations reduce to previously known versions in special cases. The equations agree with other work on random radiation fluids when the scalar field is absent and $\delta Q^{(\phi)} = \delta J^{(\phi)} = s_{,\phi} = 0$ [42]. The non-viscous case $\eta_s = \eta_b = 0$ has been widely discussed in the context warm inflation [8, 51]. A new feature of these equations is the noise term in the energy and momentum transfer terms (3.27) and (3.28), and the effect on the amplitude of density perturbations will be analyzed later. The viscous case without the random fluid sources has been discussed in [44, 45].

C. Scalar equation

The perturbed version of the Langevin equation of the scalar field is constructed along similar lines. We employ the equivalence principle to infer the curved space Langevin equation

$$-\square\phi(x, t) + \Upsilon D\phi(x, t) + \Omega_{,\phi} = (2\Upsilon T)^{1/2}\xi^{(\phi)}(x, t), \quad (3.29)$$

After replacing the scalar field by $\phi + \delta\phi$ and using the perturbed metric (3.1) to get the first order perturbation equation

$$(\delta\dot{\phi} - \alpha\dot{\phi})^\cdot + 3H(\delta\dot{\phi} - \alpha\dot{\phi}) - \nabla^2\delta\phi + \Omega_{,\phi\phi}\delta\phi - \kappa\dot{\phi} + \delta\Upsilon\dot{\phi} + \Upsilon\delta\dot{\phi} - \alpha\ddot{\phi} = (2\Upsilon T)^{1/2}\xi^{(\phi)}. \quad (3.30)$$

The correlation function for the stochastic sources has to be corrected for the shift to comoving coordinates, as we did for the fluid,

$$\langle \xi^{(\phi)}(x, t)\xi^{(\phi)}(x', t') \rangle = a^{-3}\delta^{(3)}(x - x')\delta(t - t'). \quad (3.31)$$

Note that we have made no changes to the noise coefficient in Eq. (3.30) compared to the one used in flat spacetime, assuming that the noise correlation functions are local and subject to the equivalence principle. However, the effective damping term in the equation is no longer Υ but $\Upsilon + 3H$. Consequently, the scalar field correlations are smaller than they would be in flat spacetime. We interpret this effect as being due to the scalar field correlations being non-local in time and sensitive to the finite timescale H^{-1} set by the expansion. As a result, if $\Upsilon \ll H$, it is possible for the thermal fluctuations in the scalar field to become smaller than the quantum fluctuations, which are at least of order H^2 .

The noise terms take no account of the quantum vacuum fluctuations and the thermal fluctuations [4] of the inflaton, and therefore the formalism has to be modified to properly describe the $\Upsilon \ll H$ limit. One way to do this through the noise terms has been described in [49], and we will use this in Sect V. Another approach is to make use of the linearity of the perturbation equations to add the vacuum fluctuations in as an initial condition. Since the quantum modes and the fluctuations both evolve by the same homogeneous equations, this will reproduce the quantum vacuum contribution correctly.

D. Gauge-invariant variables

After choosing a gauge, Eqs. (3.24), (3.25) and (3.30) can be combined with any two of the Einstein equations (3.3) or (3.6-3.8) to form a complete system. For example, in constant-curvature gauge $\varphi = 0$, denoted by a subscript φ , Eqs. (3.3), (3.6) and (3.7) imply

$$H\alpha_\varphi = -4\pi G(\rho + p)\delta v_\varphi, \quad (3.32)$$

$$H\kappa_\varphi = -4\pi G\delta\rho_\varphi, \quad (3.33)$$

These can be used together with Eqs. (3.24), (3.25) and (3.30) for the fluid and scalar perturbations. Alternatively, in constant-shear gauge $\chi = 0$, denoted by a subscript χ , Eqs. (3.7) and (3.8) imply

$$\kappa_\chi = -12\pi G(\rho + p)\delta v_\chi, \quad (3.34)$$

$$\alpha_\chi = -\varphi_\chi - 8\pi G\delta\Pi. \quad (3.35)$$

The noise term appears in constant-shear gauge through the (gauge-invariant) total stress term $\delta\Pi = \delta\Pi^{(f)}$. Comparison of Eq (3.18) with the definitions (3.5) and (3.22) gives

$$\delta\Pi = -\eta_s \delta v_\chi - \frac{3}{4}(2\eta'T)^{1/2}\nabla^{-2}\xi^{(f)}, \quad (3.36)$$

where ∇^{-2} is the inverse Laplacian. In constant-shear gauge we can use Eqs. (3.6), (3.7), (3.24) and (3.30) to solve for the curvature φ_χ , density and scalar perturbations.

Whatever gauge we choose for solving the equations, the density perturbations can be expressed in terms of commonly used gauge-invariant combinations. Two popular choices are the Lukash variable Φ and the Bardeen variable Ψ [50],

$$\Phi = \varphi + H\delta v, \quad \Psi = \varphi + \frac{\delta\rho}{3(\rho+p)}. \quad (3.37)$$

We can regard these as the curvature fluctuation φ_v in comoving gauge $\delta v = 0$ and the curvature fluctuation $\varphi_{\delta\rho}$ in constant density gauge $\delta\rho = 0$, respectively. The large-scale behavior of the Lukash and Bardeen variables will play an important role in the following sections, and so we will review the large-scale behavior next.

Using the comoving wave number k , we introduce a parameter $z = k/(aH)$ which is small in the large-scale limit. The Fourier components of $\nabla^2\varphi$ and $\nabla^2\chi$ in comoving gauge are assumed to be of order z^2 . The Einstein equations (3.7) and (3.6) imply that $\kappa_v = \mathcal{O}(z^2)$ and $\delta\rho_v = \mathcal{O}(z^2)$. Gauge invariance allows us to rewrite Eqs. (3.37) in comoving gauge, so that on large scales

$$\Psi = \varphi_v + \frac{\delta\rho_v}{3(\rho+p)} \approx \Phi. \quad (3.38)$$

Furthermore, the definition (3.3) and the Einstein equation (3.9) imply

$$\dot{\Phi} = \dot{\varphi}_v \approx H\alpha_v \approx \frac{3H\delta p_v}{\rho+p} \approx \frac{3He}{\rho+p}, \quad (3.39)$$

where $e = \delta p - \bar{c}_s^2 \delta\rho$ is the gauge invariant entropy perturbation and $\bar{c}_s^2 = \dot{p}/\dot{\rho}$. In the absence of entropy perturbations, we recover the well-known result that the Lukash and Bardeen variables approach a common constant value. The random fluid can affect the large scale behavior through the generation of entropy fluctuations. The noise term in the fluid equations is suppressed by the scale factor in the correlation function (3.26). This is very convenient, because the noise and damping terms depend on physical processes which cannot apply on length scales larger than the horizon size. Note, however, that whilst the scalar field will always generate entropy as it decays, it does not necessarily generate entropy fluctuations. An example of this is warm inflation. In homogeneous warm inflationary models, the total density and total pressure are determined by a slow roll approximation in terms of the value of the scalar inflaton field. On large scales, when spatial derivatives are dropped, this is still valid and the pressure can therefore be expressed as a function of the density, consequently $\bar{c}_s^2 = \partial p/\partial\rho$ and $e = 0$.

IV. DISSIPATION AND VISCOSITY COEFFICIENTS

In this section we give generic expressions for the dissipation and viscosity coefficients, which can be derived once a specific model for the scalar field and its coupling to other fields is specified.

A. The Langevin equation of motion for the scalar field and the dissipation coefficient

The derivation of the Langevin equation of motion for the scalar field eq. (2.6) typically starts by setting all fields in the context of the so called in-in, or Schwinger closed-time path functional formalism [52]. In this formalism time dependence of the quantities and nonequilibrium evolution can be properly described. In particular, the dissipation and the stochastic noise source terms appearing in eq. (2.6) can be both determined. In the closed-time path formalism the time integration is along a contour in the complex time plane, going from $t = -\infty$ to $+\infty$ (forward branch) and

then back to $t = -\infty$ (backwards branch). Fields are then identified on each of the time branches like, e.g., ϕ_1 and ϕ_2 , respectively, and so on for all other fields in the system. Due to the duplication of field variables in this formalism, four two-point Green functions can be constructed with each of these fields. In addition, it is also convenient in this formalism to work in a rotated basis for the field variables, called the Keldshy basis, where we define new field variables: $\phi_c = (\phi_1 + \phi_2)/2$ and $\phi_\Delta = \phi_1 - \phi_2$. The effective equation of motion for the scalar field is obtained from the saddle point equation [35]

$$\left. \frac{\delta \Gamma[\phi_c, \phi_\Delta]}{\delta \phi_\Delta} \right|_{\phi_\Delta=0} = 0, \quad (4.1)$$

where $\Gamma[\phi_c, \phi_\Delta]$ is the effective action for the scalar field. This leads to an effective Langevin-like stochastic equation of motion of the form [12, 35]

$$\int d^4 x' \mathcal{O}_R[\phi_c](x, x') \phi_c(x') = \xi(x), \quad (4.2)$$

where $\mathcal{O}_R[\phi_c](x, x')$ is defined as

$$\mathcal{O}_R[\phi_c](x, x') = [\partial^2 + V''(\phi_c) + \Sigma_{R,\text{local}}] \delta^4(x - x') + \Sigma_R[\phi_c](x, x'), \quad (4.3)$$

where $\Sigma_R[\phi_c](x, x')$ are retarded corrections coming from the functional integration that leads to the effective action in the Keldshy basis and $\Sigma_{R,\text{local}}$ indicates local corrections. The term on the right hand side in eq. (4.2), can be interpreted as a Gaussian stochastic noise with the general properties of having zero mean, $\langle \xi(x) \rangle = 0$, and two-point correlation

$$\langle \xi(x) \xi(x') \rangle = \Sigma_F[\phi_c](x, x'), \quad (4.4)$$

where $\Sigma_F[\phi_c](x, x')$ is the diagonal self-energy in the Keldshy basis.

Equation (4.2) is a nonlocal, non-Markovian equation of motion for ϕ . It can be shown though that when there is a clear separation of timescales in the system, this equation can be well approximated by a local, Markovian approximation [36]. In this case, the eqs. (4.2) and (4.4) becomes of the form of eqs. (2.6) and (2.7), with a local dissipation coefficient that is defined by [12, 53]

$$\Upsilon = \int d^4 x' \Sigma_R[\phi_c](x, x') (t' - t). \quad (4.5)$$

For example, if we consider an interaction term of the scalar field ϕ with some other scalar field χ as given by $(g^2/4)\phi^2\chi^2$, then the $\Sigma_R[\phi_c](x, x')$ term in eq. (4.5) becomes

$$\Sigma_R[\phi_c](x, x') = -ig^4\phi_c^2 \theta(t - t') \langle [\chi^2(x), \chi^2(x')] \rangle. \quad (4.6)$$

Complete expressions for Υ can be found, e.g., in ref. [54, 55] for different interactions and regimes of parameters.

B. The viscosity coefficients

The shear and bulk viscosity coefficients have been computed in previous works and defined through Kubo formulas [56], which are derived in the context of linear response theory (see also [57]):

$$\eta_s = \frac{1}{20} \lim_{\omega \rightarrow 0} \frac{1}{\omega} \int d^3 x dt e^{i\omega t} \langle [\Pi_{lm}(\mathbf{x}, t), \Pi^{lm}(0)] \rangle, \quad (4.7)$$

$$\eta_b = \frac{1}{2} \lim_{\omega \rightarrow 0} \frac{1}{\omega} \int d^3 x dt e^{i\omega t} \langle [\mathcal{P}(\mathbf{x}, t), \mathcal{P}(0)] \rangle, \quad (4.8)$$

where

$$\Pi_{lm}(x) = T_{lm}(x) - \frac{1}{3}\delta_{lm}T_i^i(x), \quad (4.9)$$

is the traceless part of the stress tensor and

$$\mathcal{P}(x) = -\frac{1}{3}T_i^i(x) + v_s^2 T_{00}(x), \quad (4.10)$$

where v_s is the local (equilibrium) speed of sound (introduced explicitly in the quantum field theory calculation for consistency, see e.g. [58, 59])

$$v_s^2 = \frac{\partial p}{\partial \rho}. \quad (4.11)$$

The averages in Eqs. (4.7) and (4.8) are again with respect to thermal equilibrium.

Typical expressions for the bulk and shear coefficients follow from the standard hydrodynamics which expresses these coefficients in terms of the collision time τ of the radiation bath and the radiation energy density ρ_r [60]

$$\eta_s = \frac{4}{15}\rho_r\tau, \quad (4.12)$$

$$\eta_b = 4\rho_r\tau\left(\frac{1}{3} - v_s^2\right)^2. \quad (4.13)$$

Note that in conformal field theories $v_s^2 = 1/3$ and the bulk viscosity vanishes identically. This is because dilatation is a symmetry and the fluid remains always in equilibrium. Likewise, for scale invariant field theories, for an ideal equation of state, $\omega_r = 1/3$, the bulk viscosity also vanishes. But quantum corrections in quantum field theories in general break scale invariance (where the renormalization group β -function is nonvanishing) and the bulk viscosity is nonvanishing as well. The bulk viscosity becomes directly proportional to the measure of breaking of scale-invariance. This is the case of standard scalar and gauge field theories in general and these are the type of field theories we consider to describe the particles in the radiation bath in a microscopic context. In any case, the bulk viscosity is expected to be smaller than the shear.

It is useful to give the viscosity coefficients for an explicitly example of quantum field theory, e.g., for a self-interacting quartic scalar field model, $\lambda_\sigma\sigma^4/4!$, where these viscosities were derived in [58]. From the results obtained in [58], the bulk viscosity for the case of the scalar quartic self-interaction in the weak interaction regime is given by

$$\eta_b \simeq \begin{cases} 5.5 \times 10^4 \frac{\tilde{m}_\sigma^4 m_\sigma^2(T)}{\lambda_\sigma^4 T^3} \ln^2 [1.2465 m_\sigma(T)/T], & m_\sigma \ll T \ll m_\sigma/\lambda_\sigma \\ 8.9 \times 10^{-5} \lambda_\sigma T^3 \ln^2(0.064736 \lambda_\sigma), & T \gg m_\sigma/\lambda_\sigma, \end{cases} \quad (4.14)$$

while the shear viscosity is the same in the two temperature regimes given in Eq. (4.14),

$$\eta_s \simeq 3.04 \times 10^3 \frac{T^3}{\lambda_\sigma^2}, \quad (4.15)$$

where, in the above expressions, $m_\sigma(T)$ is the σ scalar field thermal mass, $m_\sigma^2(T) = m_\sigma^2 + \lambda_\sigma T^2/24 [1 + \mathcal{O}(m_\sigma/T)]$, $\tilde{m}_\sigma^2 = m_\sigma^2(T) - T^2(\partial m_\sigma^2(T)/\partial T^2) \simeq m_\sigma^2 - \beta(\lambda_\sigma)T^2/48$, where $\beta(\lambda_\sigma) = 3\lambda_\sigma^2/(16\pi^2)$ is the renormalization group β -function.

C. Perturbations of the Dissipative and Viscosity Coefficients

To complete the specification of the fluctuation equations, we need $\delta\Upsilon$ and $\delta\eta_b$, the fluctuations of the dissipation and bulk viscosity coefficient. For a general temperature T and field ϕ dependent dissipative coefficient, given by

$$\Upsilon = C_\phi \frac{T^c}{\phi^{c-1}}, \quad (4.16)$$

we obtain that

$$\delta\Upsilon = \Upsilon \left[c \frac{\delta T}{T} - (c-1) \frac{\delta\phi}{\phi} \right]. \quad (4.17)$$

Likewise, the quantum field derivations for the bulk and shear viscosity coefficients, η_b and η_s , respectively, show that they can be parametrized in the form

$$\eta_b = C_b T^d / m_r^{d-3}, \quad (4.18)$$

$$\eta_s = C_s T^s / m_r^{s-3}, \quad (4.19)$$

where m_r is just a constant mass scale (typically the renormalized bare mass for the particles in the radiation bath, for example, from the expressions in Subsec. IV B, $m_r \equiv m_\sigma$). The temperature exponents d and s for the bulk and the shear viscosity coefficients are given by the specific quantum field theory model realization describing the particles in the thermal bath and the specific parameter regime in study. For example, from the expressions (4.14) and (4.15) for the viscosity coefficients derived from a thermal $\lambda_\sigma \sigma^4$ scalar field model, which is the relevant case for warm inflation model building, we have $d = s = 3$ in the high temperature regime $T \gg m_\sigma / \lambda_\sigma$. In this work we will be working with this temperature dependence for both the bulk and shear viscosity coefficients.

As far the perturbations are concerned, we should also note that a bulk viscous pressure is a background quantity, while a shear viscous pressure is a perturbation quantity originating from momentum perturbations. Thus, we only need to account for perturbations of the bulk viscous pressure. From Eq. (4.18), the perturbation of the bulk viscosity, $\delta\eta_b$, we have, similarly as for the dissipation coefficient, that

$$\frac{\delta\eta_b}{\eta_b} = d \frac{\delta T}{T}. \quad (4.20)$$

Although dissipation implies departures from thermal equilibrium in the radiation fluid, the system has to be close-to-equilibrium for the calculation of the dissipative coefficient to hold, therefore we assume $p_r \simeq \rho_r/3$, *i.e.* we consider $\omega_r = 1/3$. Using $\omega_r = 1/3$, then we have that $\rho_r \propto T^4$. Thus, δT appearing in Eqs. (4.17) and (4.20), can be expressed in terms of the radiation energy density and its perturbation as

$$\frac{\delta T}{T} \simeq \frac{1}{4} \frac{\delta\rho_r}{\rho_r}. \quad (4.21)$$

V. COSMOLOGICAL PERTURBATIONS DURING INFLATION

In warm inflation [5, 61] (for earlier related work, see for instance [3, 62]) there is a non-negligible contribution from the radiation bath to the power spectrum. The radiation bath originates from the decay of fields coupled to the inflaton field and triggered by its dynamics during inflation. Thus, during warm inflation there is a two-component fluid made of a mixture of the scalar inflaton field interacting with the radiation fluid, due to the produced decay particles. Thus, density fluctuations are sourced primarily by thermal fluctuations of the inflaton field when coupled with the thermal radiation bath. These are modeled by the stochastic Langevin equation Eq. (3.30) for the inflaton ϕ , with dissipative and stochastic noise terms satisfying a fluctuation and dissipation relation. In fact, this stochastic equation for the inflaton field can be completely derived from first principles, as shown *e.g.* in [35, 36, 70]. It has also been shown in [65, 71] that it appropriately governs the evolution of the inflaton perturbations during warm inflation.

Explicit microscopic derivations of the resulting dissipation term in the inflaton effective dynamical evolution equation show that the resulting two-fluid system in warm inflation is coupled [12, 63, 64]. This happens because both fluid components can exchange energy and momentum through the dissipation term. As was shown in [65], the temperature dependence of the dissipation coefficient causes a coupling between the inflaton perturbations with those of the radiation perturbations. This effectively leads to growing modes in the power spectrum, which can cause considerable fine-tuning of the inflaton potential parameters for warm inflation. These growing modes get worse the larger is the power in temperature in the dissipation term [44, 65]. Earlier work had developed the expression for the primordial spectrum in warm inflation [8, 51], but had not accounted for the growing mode.

There can also be other intrinsic microscopic decay processes in the produced radiation bath, causing it to depart from equilibrium. These intrinsic dissipative effects in the radiation fluid will cause it deviated from a perfect fluid

during inflation. As the radiation fluid departs from equilibrium, pressure and momentum changes are produced by the particle excitations and this generates viscous effects. Among these are the bulk and shear viscous pressures. The presence of these viscous processes during warm inflation can control the growing mode arising from the temperature dependent dissipative coefficient, as has been recently studied in [44].

In [44] the fluctuation spectrum in warm inflation was studied by including only the effects of shear viscosity in the radiation fluid and it was assumed that the bulk viscosity is much smaller than the shear viscosity. This is the case for quasi-conformal radiation fluids. For instance, in quantum field theory calculations in general, e.g., in perturbative quantum chromodynamics, which corresponds to the high-temperature quark-gluon phase in the early universe, the bulk viscosity has been estimate to be a factor 10^{-3} to 10^{-8} smaller than the shear viscosity [66]). Even though bulk viscosities in most regimes have smaller magnitudes than shear viscosities, there are regimes of temperature and field parameters where the bulk viscosity can be important. For instance, close to phase transitions or phase changes in general, it has been shown that the bulk viscosity can be much larger in magnitude than the shear viscosity [67]. Furthermore, the bulk viscosity, been related to pressure fluctuations, already contributes at the background level, while the shear viscosity, been related to momentum fluctuations, contribute only at the perturbation level. The effect of the bulk viscosity in warm inflation has been studied previously in [68, 69]. These papers did not treat shear viscosity effects, only looked at the case of constant dissipation (thus there was no coupling of the radiation bath perturbations with those of the field), and looked at constant bulk viscous pressure or one proportional to the radiation energy density. Under these simplifying assumptions, it was found in [68, 69] that bulk viscous effects could induce a variation in the power spectrum amplitude in the order of 4%. However, by including the full temperature dependence for both the dissipation and bulk viscosity terms, as motivated by microscopic quantum field derivations, it is possible that the effect of the bulk viscous pressure on the power spectrum can be significantly higher. This possibility will be analyzed here, where both bulk and shear viscous effects are included.

However, because of the random noise terms in the radiation fluid equations, the growing mode is not completely eliminated. In fact, we show that the random noise caused by the viscous radiation fluid tends to further contribute to the curvature perturbation spectrum, causing it to increase even in the weak dissipative regime of warm inflation (when the inflation dissipation term is smaller than the Hubble parameter). Besides, viscous random noise terms tend to add more power on smaller scales, rendering the primordial spectrum blue-tilted. Since the random fluctuation contributions that are added to the curvature perturbation spectrum are proportional to the viscosity coefficients, this allow us to put strong constraints on the level of viscosity allowed in the warm inflation scenario.

A. Primordial spectrum and spectral index

In warm inflation, the scalar field ϕ is an inflaton and the energy density is dominated by a temperature independent potential $V(\phi)$. The inflaton decay is described by the damping coefficient $\Upsilon(\phi, T)$. The radiation fluid $\rho_r \equiv \rho^{(f)}$ is produced, and continually replenished by decay of the inflaton field. The background fields satisfy

$$\ddot{\phi} + (3H + \Upsilon)\dot{\phi} + V_{,\phi} = 0, \quad (5.1)$$

$$\dot{\rho}_r + 4H \left(\rho_r - \frac{9}{4} H \eta_b \right) = \Upsilon \dot{\phi}^2, \quad (5.2)$$

$$3H^2 = 8\pi G \rho. \quad (5.3)$$

Prolonged inflation requires the slow-roll conditions $|\epsilon_X| \ll 1$, where $\epsilon_X = -d \ln X / H dt$, and X is any of the background field quantities. The background equations at leading order in the slow-roll approximation of small ϵ_X become

$$3H(1 + Q)\dot{\phi} \simeq -V_{,\phi}, \quad (5.4)$$

$$4\rho_r \simeq 3Q\dot{\phi}^2 + 9H\eta_b, \quad (5.5)$$

$$3H^2 \simeq 8\pi G V, \quad (5.6)$$

where $Q = \Upsilon/(3H)$.

We are interested in deriving the effect with a variable background on the amplitude of the spectrum and its spectral index n_s . Numerically, we have integrated Eqs. (3.24), (3.25) and (3.30) for a set of modes for different wavenumbers, together with the background equations (5.1)-(5.3). For the background evolution, we consider a quartic chaotic model with inflationary potential $V = \lambda\phi^4/4$. For the dissipative parameter, we focus on a cubic dependence with the temperature, $c = 3$, $\Upsilon = C_\phi \frac{T^3}{\phi^2}$, and similarly for the shear and bulk viscosities, with $\eta_s \propto T^3$ and $\eta_b \propto T^3$. This is the dependence obtained when dissipation is given by the decay into light degrees of freedom of a scalar massive field

coupled to the inflaton, with mass $m \simeq g\phi$ [53–55]. Note that the dissipation and viscosity coefficients quoted here can be used to a good approximation for ϕ and radiation fluid modes $k \lesssim T$. As k approaches up to T , there will be corrections to these quantities that can be computed [36] and for $k > T$ these coefficients will decrease exponentially.

For the metric perturbations, we work in the zero-shear gauge, $\chi = 0$. Using the slow-roll equations (5.4)-(5.6), the gauge-invariant Lukash variable Φ defined in Eq. (3.37) is now

$$\Phi = -\frac{1}{1+Q}\zeta^\phi - \frac{Q}{1+Q}\zeta^v, \quad (5.7)$$

where

$$\zeta^\phi = -\varphi + H\delta\phi/\dot{\phi}, \quad (5.8)$$

$$\zeta^v = -\varphi - \delta v_r. \quad (5.9)$$

At late times, when $z \rightarrow 0$, we have $\Phi = -\zeta^\phi = -\zeta^v$. The power spectrum is given by:

$$\langle \zeta^i(k, t) \zeta^i(k', t) \rangle = P^i(k, t) (2\pi)^3 \delta^{(3)}(k + k'), \quad (5.10)$$

which can then be used to obtain the power spectrum of the gauge-invariant comoving curvature perturbations $P_{\mathcal{R}}(k) = (4\pi)k^3 P_\Phi(k)$. The quantum statistical average is the same as the stochastic average in our formalism. Since the gauge-invariant perturbations are constant on large scales, $P_{\mathcal{R}}(k, t)$ will approach a constant value $P_{\mathcal{R}}(k)$, which we identify as the primordial amplitude of density perturbations [44]. We normalise the system in a periodic box of size l to replace the momentum delta-function $\delta^{(3)}(0)$ by l^3 . The variables can be re-scaled to absorb the factor l^3 by defining

$$\bar{\zeta}^i = (k/2\pi l)^{3/2} \zeta^i, \quad \bar{\xi}^i = (k/2\pi l)^{3/2} \hat{\xi}^i. \quad (5.11)$$

As a result, $\bar{\xi}^i$ is a unit normalized random variable with variance

$$\langle \bar{\xi}^i(t) \bar{\xi}^i(t') \rangle = \delta(t - t'). \quad (5.12)$$

Once the stochastic equations are solved, the power spectra are given by

$$k^3 P^i(k, t) = \langle \bar{\zeta}^i(k, t) \bar{\zeta}^i(k, t) \rangle. \quad (5.13)$$

There is no residual dependence on the normalisation scale l .

In the chaotic quartic model, background evolution is such that the dissipative ratio Q and T/H increase during inflation, and inflation ends when the slow-roll conditions are violated [13, 72, 73]. Radiation is given as usual by $\rho_r = \pi^2 g_* T^4/30$, and for the number of relativistic degrees of freedom we take $g_* = 15/4$. For the smaller k mode considered, we set horizon crossing at 50 e-folds before the end of inflation. For this parameter value, we have $Q_* \simeq 10^{-7}$ as the lower value consistent with the condition $T/H \geq 1$. But even for such a low Q_* value, by the end of inflation we have $Q_{50} > 1$. We have run the simulations with $\lambda = 10^{-14}$. The amplitude of the primordial spectrum can be normalized to the Planck value [74], $P_{\mathcal{R}}^{1/2} = 4.69 \times 10^{-5}$ by slightly adjusting the value of λ , but this will have little effect on the spectral index.

Inflaton thermal and quantum fluctuations, relevant in the very weak dissipative regime $Q_* \ll 1$, are taken into account by adding another stochastic noise term $\xi^{(q)}$ to the field Langevin equation, as described in [49]. Dissipative processes may maintain a non-trivial distribution of inflation particles which for sufficiently fast interactions should approach the Bose-Einstein distribution $n_{BE}(k) = (e^{k/aT} - 1)^{-1}$. Both possibilities, either negligible inflaton occupation number at horizon crossing $N_* \simeq 0$ or given by a thermal distribution, will be considered by adding the following stochastic term to the field equation (3.30):

$$H \frac{\sqrt{1+2N_*}}{\sqrt{2}} \xi^{(q)}, \quad (5.14)$$

with the same correlation function for the stochastic noise than that of $\xi^{(\phi)}$ in Eq. (3.30). In addition, we have numerically explored the two possibilities encountered in section II when discussing the mixture of a relativistic fluid and a scalar field in a cosmological set-up: having either the field stochastic noise $\xi^{(\phi)}$ in the energy flux ($C_P = 0$) or in the momentum flux ($C_P = 1$). We focus mainly on low values of the dissipative ratio at horizon crossing $Q_* \lesssim 10$. As we will see, in this regime the different interplay of the stochastic terms in both fluid and field equations makes a

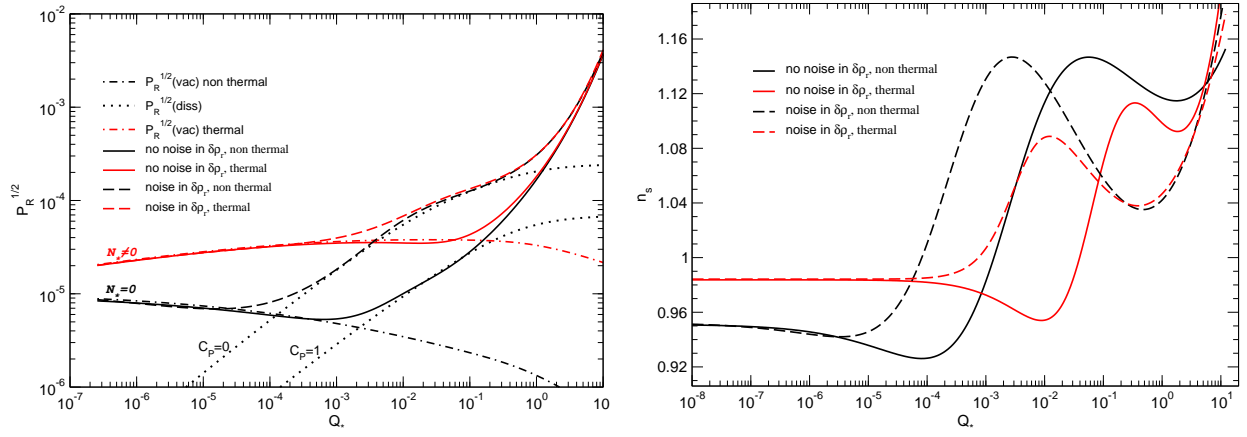


FIG. 1: LHS plot: Amplitude of the primordial spectrum versus Q_* . Solid (dashed) lines have been obtained with $C_P = 1$ (0); black lines are for $N_* = 0$ (non-thermal) and gray (red) lines for $N_* = n_{BE}$ (thermal). For comparison, the dotted line shows the analytical solution for $P_{\mathcal{R},diss}^{1/2}$, and dot-dashed lines $P_{\mathcal{R},vac}$. The amplitude of the spectrum becomes constant some e-folds after horizon crossing, and here we are plotting the value at $N_e = 20$. The total number of e-folds is 50. RHS plot: spectral index versus Q_* . We have taken $\lambda = 10^{-14}$.

difference in the spectrum and the spectral index. For larger values of Q_* , the effect of the growing mode observed in [65] dominates the spectrum.

For the dissipative coefficient in the weak regime, $Q_* \ll 1$, the inflaton is effectively decoupled from the radiation fluid, and the analytical expression for the amplitude of the primordial spectrum is given by [49, 73]:

$$P_{\mathcal{R}} = (P_{\mathcal{R},diss} + P_{\mathcal{R},vac}) = \left(\frac{H_*}{\dot{\phi}_*}\right)^2 \left(\frac{H_*}{2\pi}\right)^2 \left[\frac{T_*}{H_*} \frac{2\pi Q_*}{\sqrt{1 + 4\pi Q_*/3}} + 1 + 2N_* \right], \quad (5.15)$$

where all variables are evaluated at horizon crossing. The first term is the direct contribution due to dissipation, the stochastic term $\xi^{(\phi)}$, while the others are due to the inflaton fluctuations including the vacuum term ($N_* = 0$) and the contribution from thermal excitations ($N_* = n_{BE}(a_* H_*)$). The analytical expression for $P_{\mathcal{R},diss}$ was obtained for the scenario with $C_P = 1$. When the radiation source term includes the dissipative stochastic noise, this enhances the amplitude of radiation fluctuations, which backreacts earlier on the inflaton fluctuations, and dissipation dominates the spectrum for smaller values of Q_* (dashed lines in the LHS plot in Fig. (1)). Numerically, up to $Q_* \simeq 0.1$, the contribution $P_{\mathcal{R},diss}$ gets enhanced in this case by a factor $\mathcal{O}(40)$,

$$P_{\mathcal{R},diss}^{C_P=0} \simeq \left(\frac{H_*}{\dot{\phi}_*}\right)^2 \left(\frac{H_*}{2\pi}\right)^2 \frac{T_*}{H_*} \frac{80\pi Q_*}{\sqrt{1 + 4\pi Q_*/3}}. \quad (5.16)$$

Taking this effect into account, for small values of $Q_* \lesssim 1$ the amplitude can be written as:

$$P_{\mathcal{R}} = \left(\frac{H_*}{\dot{\phi}_*}\right)^2 \left(\frac{H_*}{2\pi}\right)^2 [\Delta_{Q_*} + 1 + 2N_*], \quad (5.17)$$

where,

$$\Delta_{Q_*} \simeq 80\pi Q_* \frac{T_*}{H_*}, \quad C_P = 0, \quad (5.18)$$

$$\Delta_{Q_*} \simeq 2\pi Q_* \frac{T_*}{H_*}, \quad C_P = 1. \quad (5.19)$$

The analytical expression of the spectrum matches the numerical values up to $Q_* \lesssim 0.1$, as can be seen on the LHS plot in Fig. (1)). For larger values, radiation back-reacts onto the inflation fluctuations and there is a “growing mode” in the spectrum, with $P_{\mathcal{R}} \propto Q_*^\alpha$. When $N_* \simeq 0$, the spectrum is dominated by the inflaton vacuum contribution upto $Q_* \simeq 0.001(10^{-4})$ when $C_P = 1$ ($C_P = 0$); after which dissipation takes over (dotted line) up to $Q_* \simeq 0.1$; whereas for a thermal inflaton with $N_* \neq 0$, the vacuum contribution is enhanced by a factor $\coth(T_*/2H_*) \simeq T_*/H_*$,

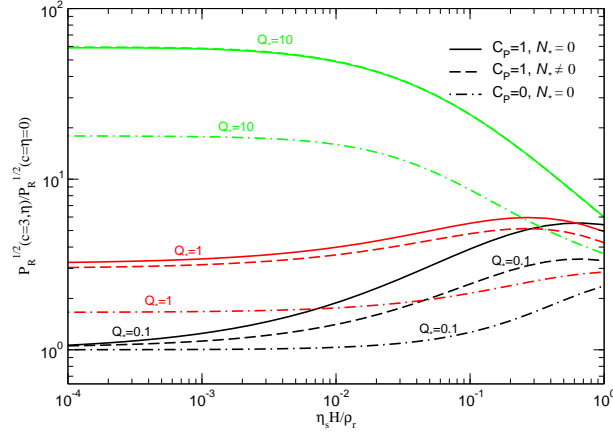


FIG. 2: Amplitude of the primordial spectrum versus $\bar{\eta}_{s*} = H_* \eta_{s*} / \rho_{r*}$, for different values of Q_* as indicated in the plot. Solid (dashed) lines have been obtained with $C_P = 1$ and $N_* = 0$ ($N_* \neq 0$); dot-dashed line with $C_P = 0$. In the later case, the statistical inflaton state does not make any difference. The amplitude of the spectrum has been normalized by its analytical value for $c = 0$.

and it dominates until larger values of $Q_* \gtrsim 0.1$. For values of $Q_* \gtrsim 10$, the inflaton statistical state does not make any difference, and the amplitude of the spectrum is fully dominated by dissipation and the induced growing mode. Moreover it does not make any difference whether or not the stochastic dissipative noise sources the radiation in the strong dissipative regime.

The spectral index is given in the companion plot in Fig. (1)). Analytically, this is given by:

$$n_s - 1 = \frac{dP_{\mathcal{R}}}{d \ln N_e} \simeq 2\eta_* - 6\epsilon_* + \frac{4N_*}{1 + 2N_* + \Delta_{Q*}}(2\epsilon_* - \eta_* + \sigma_*) + \frac{2\Delta_{Q*}}{1 + 2N_* + \Delta_{Q*}}(7\epsilon_* - 4\eta_* + 5\sigma_*), \quad (5.20)$$

where

$$\epsilon = -\frac{1}{H} \frac{d \ln H}{dt} \simeq \frac{m_P^2}{2(1+Q)} \left(\frac{V_{,\phi}}{V} \right)^2, \quad \sigma = -\frac{1}{H} \frac{d \ln \phi}{dt} \simeq \frac{m_P^2}{1+Q} \frac{V_{,\phi}/\phi}{V}, \quad \eta = -\frac{1}{H} \frac{d \ln V_{,\phi}}{dt} \simeq \frac{m_P^2}{1+Q} \frac{V_{,\phi\phi}}{V} \quad (5.21)$$

In particular for the quartic potential, they are given by:

$$\epsilon = \frac{2}{3}\eta = 2\sigma = 8 \left(\frac{m_P}{\phi} \right)^2 \frac{1}{1+Q}, \quad (5.22)$$

and the spectral index when $Q_* \ll 1$ reads:

$$n_s - 1 = -3\epsilon_* + \frac{4N_*}{1 + 2N_* + \Delta_{Q*}}\epsilon_* + \frac{7\Delta_{Q*}}{1 + 2N_* + \Delta_{Q*}}\epsilon_*. \quad (5.23)$$

Therefore, in the very weak dissipative regime with $\Delta_{Q*} \ll 1$, we have a red-tilted spectrum:

$$n_s \simeq 1 - 3\epsilon_*, \quad (N_* \simeq 0), \quad (5.24)$$

$$n_s \simeq 1 - \epsilon_*, \quad (N_* \simeq n_{BE}), \quad (5.25)$$

whereas when $\Delta_{Q*} \gtrsim 1 + 2N_*$ the spectrum turns blue $n_s \simeq 1 + 4\epsilon_*$. Again, this happens earlier when $C_P = 0$ (dashed lines in Fig. (1)). Up to that point, the spectral index is consistent with Planck values. When $C_P = 1$ and the stochastic noise does not source the radiation energy density fluctuations (solid lines), the spectral index decreases before the Δ_{Q*} contribution becomes non-negligible: this is due to the evolution of Q during inflation in this model, always increasing. Even when small at horizon crossing, it will become larger than one before the end of inflation, and dissipation will dominate over the Hubble friction. We have then 50 e-folds of inflation for smaller values of the inflaton field, i.e., larger values of ϵ_* , and thus a slightly more red-tilted spectrum. Soon after, dissipation (Δ_{Q*}) takes over and the spectrum quickly becomes blue-tilted.

We now turn to the effect of the viscosities on the spectrum. We first set the bulk viscosity η_b to zero, and consider the effects of shear. In Fig. (2) we show the primordial spectrum normalized by its analytical value when $c = 0$ (Eq.

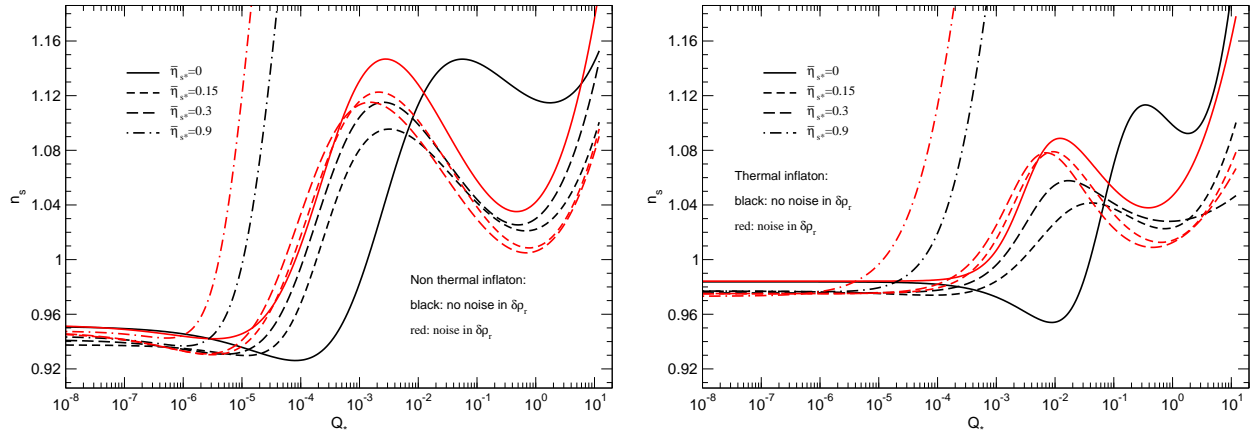


FIG. 3: RHS plot: spectral index versus Q_* , with a non-thermal inflaton, for different values of the shear parameter $\bar{\eta}_{s*} = \eta_{s*} H_*/\rho_{r*}$ at horizon crossing. Black lines do not include any noise term in the radiation fluctuations ($C_P = 1$), while red lines do so ($C_P = 0$). LHS plot: same for a thermal inflaton. We have taken $\lambda = 10^{-14}$, and $N_* = 50$.

5.17) as a function of the shear parameter $\bar{\eta}_{s*} = H_* \eta_{s*}/(\rho_{r*})$. Viscous effects will tend to damp down the effect of the growing mode, although only for values $Q_* \gtrsim \mathcal{O}(1)$; and the growing mode will only effectively disappear for values of the shear parameter $H_* \eta_{s*}/\rho_{r*} > 1$, beyond the limit of validity of the assumption of being close-to-equilibrium. For values $Q_* \leq 1$, indeed the amplitude gets enhanced. This is due to the stochastic noise term in the momentum fluid equation due to viscosity, which sources the fluid momentum perturbations. This effect dominates over the friction effect introduced by the viscosity, renders the amplitude larger, and through the fluid energy density fluctuations will in turn affect the field equations.

In Fig. (3) we have considered the effects of the shear viscosity in the spectral index, for different values of the shear parameter at horizon crossing. For values $Q_* \lesssim 1$, shear will also set more power on larger wavenumbers, implying a blue-tilted spectrum. The stochastic shear noise effect is similar to that of the field noise when included in the energy flux, rendering the spectrum consistent with Planck data only in the very weak dissipative regime $Q_* \ll 1$.

In Fig. (4) we show the total power spectrum as a function of only bulk viscosity and with a combination of bulk and shear viscosities. The results are for the case of thermalized inflaton perturbations and for the radiation noise term ($C_P = 0$). Other cases produce results that are quantitatively not much different than the ones shown. It is noticed that the larger is the dissipation coefficient, the larger the amplitude of the power spectrum gets with respect to the case where it is insensitive to magnitude of Q , given by $c = 0$, i.e., a temperature independent dissipation coefficient. This is the growing mode resulting from the coupling of the inflaton and radiation perturbations as found in [65] and also studied in [44], where also the effects of shear viscosity were considered. From Fig. 4 we can notice that the combination of bulk and shear viscosities tend to damp the spectrum quicker than including only bulk viscosity. But the spectrum only gets effectively damped to the values where the growing mode is compensated for values of the bulk and or shear viscosities that are already too close to the limit of validity of the assumption of small departures from equilibrium, i.e., $\eta H/\rho_r \ll 1$.

Results for the spectral tilt for some representative values of dissipation coefficient and bulk viscosity are presented in Table I. We have included both the cases of including the radiation noise term ($C_P = 0$) and in the absence of it ($C_P = 1$) in the perturbation equation. We have included also the cases of thermal and nonthermal inflaton fluctuations for comparison. We find that in general, for not too small dissipation coefficient, $Q_* \lesssim 10^{-3}$, for thermalized inflaton fluctuations and by including a small viscosity coefficient $\eta_{b*} H_*/\rho_{r*} \lesssim 0.035$, the results can be rendered compatible with Planck data.

As the bulk viscosity increases, $\eta_{b*} H_*/\rho_{r*} \gtrsim 0.18$, the power spectrum tilt quickly increases and tends to become blue in all cases of dissipation coefficients analyzed. This indicates that the bulk viscosity coefficient cannot be larger than around $\eta_{b*} H_*/\rho_{r*} \simeq 0.18$, setting, thus an upper bound for the value of the bulk viscosity. This result is similar to that observed only with the shear viscosity, $\eta_{s*} H_*/\rho_{r*} \lesssim 0.3$, for $Q_* \gtrsim 10^{-4}$.

In summary, the dissipative stochastic forces we have in the description of the relativistic fluid will always tend to enhance the amplitude of the fluctuations in the fluid, the effect being larger on smaller scales. The effect propagates to the field (inflaton) fluctuations, with the corresponding enhancement of the amplitude of the primordial spectrum. When the evolution of the background is such that Q increases during inflation, the spectrum will tend to be blue tilted: larger wavenumbers cross the horizon at larger values of Q_* for which the effect is more pronounced. Viscous

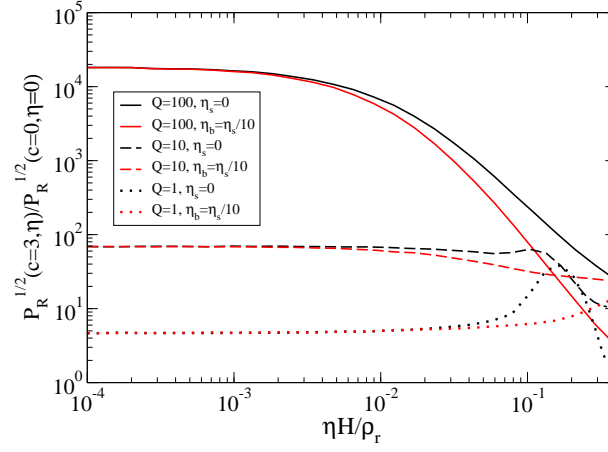


FIG. 4: The total (square root) amplitude for the power spectrum for the case $c = 3$ and normalized by its value for $c = 0$, as a function of viscosity. Three different values for the dissipation ratio Q are used to illustrate the ability of the viscosity to damp the spectrum. Black curves include only the effect of the bulk viscosity ($\eta_s = 0$) and the red includes the effects of both the shear viscosity and bulk viscosity coefficient given by $\eta_b = \eta_s/10$.

Q_*	10^{-5}			10^{-3}	
$\bar{\eta}_{b*}$	2.2×10^{-5}	0.1	0.188	0.035	0.21
(a)	0.97 ± 0.02	0.980 ± 0.001	1.008 ± 0.003	0.969 ± 0.004	1.89 ± 0.01
(b)	0.98 ± 0.01	0.989 ± 0.008	1.51 ± 0.01	1.03 ± 0.01	2.10 ± 0.01
(c)	0.97 ± 0.02	0.981 ± 0.001	0.999 ± 0.007	0.949 ± 0.006	1.73 ± 0.01
(d)	0.98 ± 0.01	0.99 ± 0.01	1.172 ± 0.009	0.973 ± 0.008	2.05 ± 0.01

TABLE I: The spectral tilt n_s for different values of Q and bulk viscosity parameter $\bar{\eta}_b = \eta_b H/\rho_r$. (a) with radiation noise term ($C_P = 0$), thermal inflaton fluctuations; (b) $C_P = 0$, non-thermal inflaton fluctuations; (c) without the radiation noise term ($C_P = 1$), thermal inflaton fluctuations; (d) $C_P = 1$, non-thermal inflaton fluctuations. These are for $N_* = 50$ and for a pivot scale of $k_0 = 1000H_0$.

effects will work in principle in the opposite direction, preventing the growth of the perturbations. However in the weak dissipative regime $Q_* < \mathcal{O}(1)$, the viscous stochastic force effect still dominates, in both the amplitude and the spectral index, sending more power to the smaller scales. Therefore, having a primordial spectrum with a spectral index consistent with Planck data will constrain the amount of viscosity (bulk and shear) allowed in the system.

Before ending this section, some comments on the tensor-to-scalar ratio r . In the weak dissipative regime this is given by:

$$r = \frac{16\epsilon_*}{\Delta_{Q_*} + 1 + 2\mathcal{N}_*}. \quad (5.26)$$

Without dissipation, when $\Delta_{Q_*} = \mathcal{N}_* = 0$, we recover the standard cold inflation result, $r = 16\epsilon_*$, and a value larger than the Planck limit $r \gtrsim 0.3$ [74], being the simplest quartic chaotic model ruled-out by observations. However dissipative dynamics, not affecting the tensors, can render r consistent with observations due to the extra suppression factor “ $\Delta_{Q_*} + 1 + 2\mathcal{N}_*$ ” [73]. Nevertheless, with negligible occupation number $\mathcal{N}_* \simeq 0$ and in the very weak dissipative regime, the suppression due to Δ_{Q_*} is not enough to render the ratio consistent with observations, as was already observed in [73]. Besides, the value of ϵ_* is slightly larger than in standard inflation because the value of the field at horizon crossing is smaller, which makes r increase initially as Q_* increases. By the time the effect of a larger Δ_{Q_*} overcomes that of ϵ_* , the spectrum has become blue-tilted. However, with non-negligible occupation number, the tensor-to-scalar ratio is always below the Planck limit $r \lesssim 0.3$, with an spectral index consistent with the data upto $Q_* \simeq 0.05$ ($Q_* \simeq 10^{-4}$) for $C_P = 1$ ($C_P = 0$).

VI. CONCLUSIONS

The matter content of the very early universe generically consists of a multi-particle system with a wide range of particle properties and interactions. Neglecting some of this richness can lead to missing out some important physical phenomena. Cold inflation is an idealization where the dynamics reduces to the classical evolution of the scalar inflaton field with vacuum quantum fluctuations superposed on this background field. Warm inflation includes additional multiparticle dynamics and recent success of its predictions [73] in fitting the Planck results provides support that these effects may have an important role to play. In this, and other situations where radiation is present in the early Universe, the idealization is often made of a perfect fluid, whereas there might be some deviations from this limit that lead to viscous dissipation and corresponding noise forces, and these effects might have observational consequences. One example where this could be applied is to the many studies looking at thermal fluctuations seeding density perturbations in a radiation dominated regime [7–11]. To provide a framework in which all these types of problems can be examined, this paper has obtained the coupled set of equations of a scalar field with dissipation interacting with an imperfect radiation fluid and treating also the corresponding density perturbations.

As an example, we have applied the equations to the warm inflation scenario, and study how the different stochastic forces affects the primordial spectrum. Einstein equations do not fully fix whether the dissipative noise source enters in the energy flux ($C_P = 0$) or in the momentum flux ($C_P = 1$), and we have explored and compared both possibilities. Previous studies of the primordial spectrum in warm inflation only took into account the second possibility with $C_P = 1$ [44, 49, 65]. It was shown that for a T dependent dissipative coefficient the amplitude of the spectrum gets enhanced for values of the inflaton dissipation term larger than the Hubble parameter, $Q_* \gtrsim 1$. The same behavior is obviously present when the noise sources directly the radiation energy density. But before the growing mode dominates the behavior of the fluctuations, the stochastic source will increase further the amplitude. In a model like the quartic potential considered here, for which Q increases during inflation, the effect is larger on smaller scales and the tilt of the spectrum increases. Nevertheless, for low enough values of $Q_* \lesssim 10^{-4}$ the effect is negligible and the spectral index remains within Planck limits. However, in order to obtain a tensor-to-scalar ratio also within the Planck upper bound, we need to consider a non-trivial (thermal) statistical distribution of inflaton fluctuations [73].

We have shown that the viscosity terms act to strongly damp the radiation perturbations in the regime where the dissipation of the inflaton field is large (compared to the Hubble parameter). Thus, the viscosities tend to counter balance the effect of the growing mode observed in [65]. However, because of the random noise terms in the radiation fluid equations, the growing mode is not completely eliminated. In fact, we also have shown that the random noise caused by the viscous radiation fluid tends to further contribute to the curvature perturbation spectrum causing it to increase even in the low dissipative regime of warm inflation (when the inflation dissipation term is smaller than the Hubble parameter). Since the random fluctuation contributions that are added to the curvature perturbation spectrum are proportional to the viscosity coefficients, for models where viscosity increases during inflation this implies more power at smaller scales, i.e., a larger tilt. And this allows us to put strong constraints on the level of viscosity permissible in the warm inflation scenario.

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